An Introduction to SMT Solvers

Johannes Kanig

INRIA, LRI, ProVal team

2 juin 2010

AdaCore

Contents

Automated Theorem Proving

Who does prove Theorems today?

- Mathematicians
- Hardware verification
- Software verification

Automation of proofs

- Makes life easier for mathematicians
- Makes it possible to state (and prove) thousands of tiny theorems
- Makes large-scale verification of hardware and software accessible and more reliable

The ProVal Team

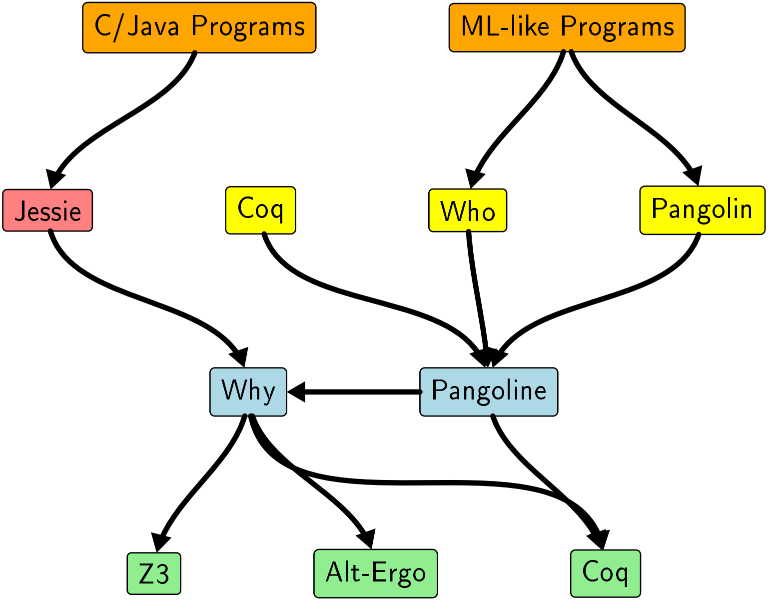
Main Topic: Program proof

- Specification languages
- Computing proof obligations (formulas that imply the correctness of the program)
- Proving proof obligations

Tools (Why Platform)

- Caduceus, Krakatoa: C/Java frontends
- Jessie: Common intermediate language
- Why: Obtain proof obligations from programs
- Alt-Ergo: Automated prover (SMT Solver)

The Why platform



A logical formula ...

$$sorted(t, i, j) =$$

$$\forall k_1, k_2 : int\ .\ i \leq k_1 \ \wedge\ k_1 \leq k_2 \ \wedge\ k_2 \leq j \Rightarrow t[k_1] \leq t[k_2]$$

. . . as seen by an SMT solver

$$sorted(t, i, j) =$$

$$\forall k_1, k_2 : int$$

$$\Downarrow$$

$$i \leq k_1 \;\wedge\; k_1 \leq k_2 \;\wedge\; k_2 \leq j \;\Rightarrow\; t[k_1] \leq t[k_2]$$

Instantiation

Logic reasoning

Theory reasoning (here: Arithmetic)

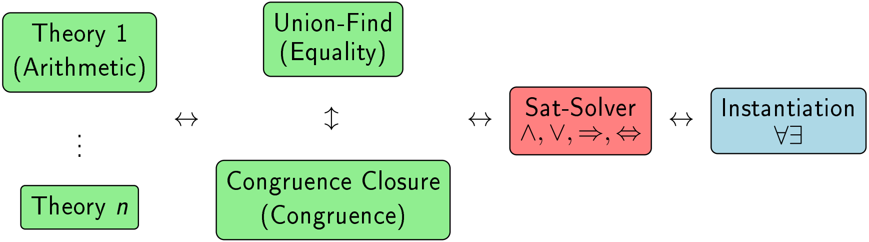
Satisfiability Modulo Theories

SMT provers divide the problem in three parts

- The theory part: equality reasoning, arithmetic reasoning, ...
- The satisfiability part: deals with logical connectors $\wedge$, $\vee$, $\Rightarrow$, $\neg$,...
- The instantiation of quantified axioms

We will look at each of the three parts in turn

The different parts of an SMT solver



A more detailed example

Hypotheses

- $H_1 : a > 0$
- $H_2 : \forall xy.x \geq y \rightarrow max(x, y) = x$

Goal

$G : f(max(a, 0)) = f(a)$

Solved by an SMT Solver (1)

Negate the Goal

$H_1 \ \wedge \ H_2 \rightarrow G$ becomes $H_1 \ \wedge \ H_2 \ \wedge \ \neg G$

Launch Sat-Solver

Assume H_1, H_2 and $\neg G$ and try to derive a contradiction

- Assume the inequality $a > 0$
- Register the lemma: $\forall xy.x \geq y \rightarrow max(x, y) = x$
- Assume the inequality $f(max(a, 0)) \neq f(a)$
- Currently no contradiction!

Instantiation

Specialize the lemma by applying it to a and 0 and replace $\rightarrow$:

$a \geq 0 \rightarrow max(a, 0) = a \qquad \Leftrightarrow \qquad a < 0 \ \vee \ max(a, 0) = a$

Solved by an SMT Solver (2)

Split the disjunction

First assume $a < 0$, then assume $\neg(a < 0)$, try to find a contradiction in both cases

Assuming $a < 0$

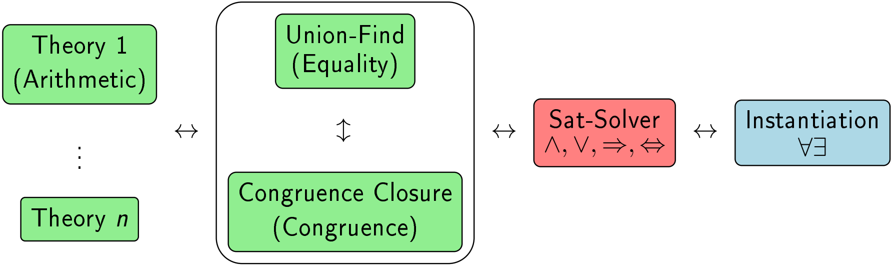
Direct contradiction with H_1 (using knowledge about the symbols $<$ and $\geq$)

Assuming $\neg(a < 0)$

- It follows $max(a, 0) = a$
- Deduce $f(max(a, 0)) = f(a)$
- Contradiction with $\neg G$

We have obtained a contradiction in all cases, the negated formula is unsatisfiable, that means the input formula is valid!

Equality Reasoning



Equality reasoning - The problem

Terms

$t ::= c \mid f(t_1, \cdots, t_n)$

Given

a list of equations $t = t'$

We want to know

Does the equation $t_1 \stackrel{?}{=} t_2$ follow?

Using the axioms

Reflexivity	$t = t$
Symmetry	$t_1 = t_2 \to t_2 = t_1$
Transitivity	$t_1 = t_2 \ \wedge \ t_2 = t_3 \to t_1 = t_3$
Congruence	$t_1 = t_2 \to f(t_1) = f(t_2)$

Example

Given

- $f^2(a) = f(f(a)) = a$
- $f^5(a) = f(f(f(f(f(a))))) = a$

We want to prove

$f(a) = a$

Proof

1. $f^5(a) = f^3(a)$ (Congruence)
2. $f^2(a) = f^3(a) = a$ (Transitivity, Symmetry)
3. $f^3(a) = f(f^2(a)) = f(a)$ (Congruence)
4. $f(a) = a$ (Transitivity of (2) and (3))

Disjoint Sets

- Goal: deal with the first three axioms efficiently
- Idea: put all terms into disjoint sets
- When two terms are in the same set, they are equal
- Initial state: every term is in his own set:

  t_1 t_2 t_3 t_4 t_5

- After treating $t_1 = t_3$ and $t_2 = t_5$:

  t_1 t_3 t_2 t_5 t_4

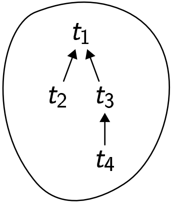
- After treating $t_1 = t_2$:

  t_1 t_2 t_3 t_5 t_4

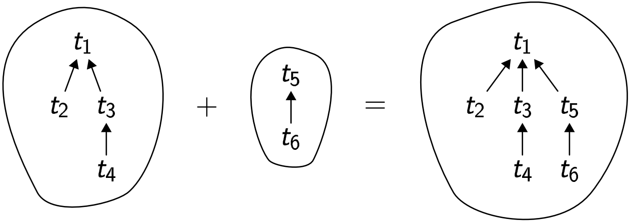
- Deciding $t \stackrel{?}{=} t'$ amounts to checking if t, t' are in the same set

Union-Find (1975)

- Represent each set by a tree with upward pointers:



- The root is the representative
- Operation find to find the representative of any term: just follow the arrows
- Operation union to treat an equality: simply point one root to the other



Two important optimizations

- Keep trees small: let point root of smaller tree to root of larger tree
- Path compression: "flatten" trees, each time we are searching for a root r starting from t, let t point directly to r afterwards
- Result: Algorithm is quasi-linear (optimal)
- Incrementality: we can add equations one by one, interleave equations $t_1 = t_2$ with queries $t_1 \stackrel{?}{=} t_2$

Inequalities $t_1 \neq t_2$

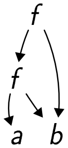
- Simply maintain the information that two sets of terms must be different
- Merging sets for which an inequality was registered leads to an inconsistency

Congruence Closure (1980)

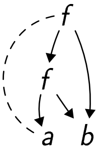
- Deal with the fourth axiom: Congruence

$$\forall xy.x = y \to f(x) = f(y)$$

  for any function symbol f
- Solution: represent a term by a directed acyclic graph (DAG) with sharing. Example: $f(f(a, b), b)$



- Add an equivalence relation to this graph (using union-find):



  represents $f(f(a, b), b) = a$

Finding new equalities

- Build a reverse dictionary mapping nodes to their fathers:

$$
\begin{array}{rcl}
a & \mapsto & f(a,b), g(a) \\
b & \mapsto & f(a,b)
\end{array}
$$

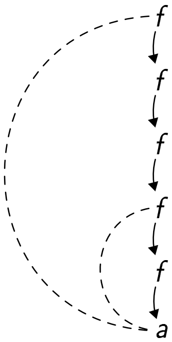
- Two new operations: find and merge.

```
merge(t₁,t₂) =
  union(t₁,t₂);
  F₁,F₂ = fathers(t₁), fathers(t₂);
  for each x in F₁, y in F₂ do
    if congruent(x,y) then merge(x,y);
  done
```

Congruence Closure — Example

Given

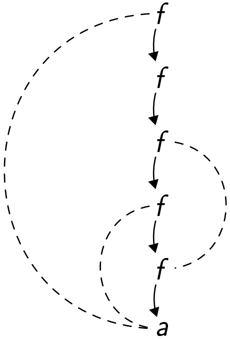
- $f^2(a) = f(f(a)) = a$
- $f^5(a) = f(f(f(f(f(a))))) = a$



Congruence Closure — Example

Given

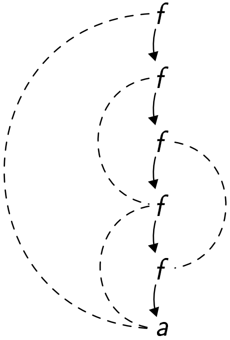
- $f^2(a) = f(f(a)) = a$
- $f^5(a) = f(f(f(f(f(a))))) = a$



Congruence Closure — Example

Given

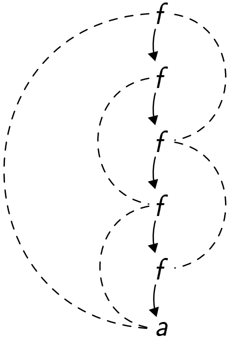
- $f^2(a) = f(f(a)) = a$
- $f^5(a) = f(f(f(f(f(a))))) = a$



Congruence Closure — Example

Given

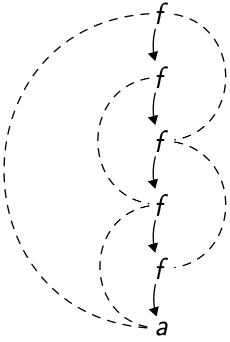
- $f^2(a) = f(f(a)) = a$
- $f^5(a) = f(f(f(f(f(a))))) = a$



Congruence Closure — Example

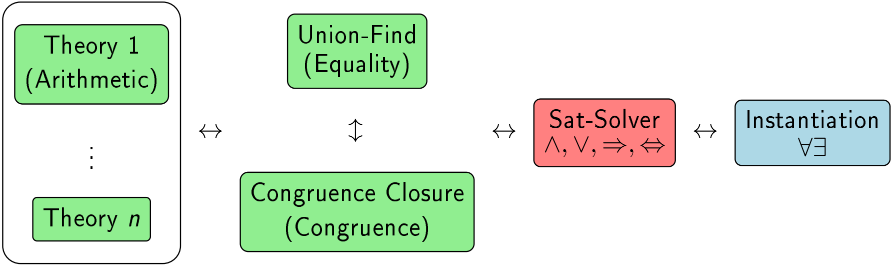
Given

- $f^2(a) = f(f(a)) = a$
- $f^5(a) = f(f(f(f(f(a))))) = a$



f(a) = a

Theory Reasoning (Arithmetic)



Arithmetic reasoning

Arithmetic

- Interprets the function symbols $+$, $-$, $\times$, $\div$, and the arithmetic constants
- But also the relation symbols $\leq$, $<$, $\geq$, $>$

There are a few algorithms to deal with Linear Arithmetic

- Gauss Elimination (Equality only)
- Fourier-Motzkin
- Simplex Algorithm

We will look more closely at these methods

Gauss Elimination

Goal: deal with equalities in linear arithmetics

- Transform term into sums of monomials: $\sum_i^k c_i t_i$
- When treating an equality between such polynomes

$$\sum_i^k c_i t_i = \sum_j^k d_i s_i$$

isolate a monomial, say, t_1, and build the equation

$$t_1 = \sum_j^k \frac{d_i}{c_1} s_i - \sum_{i \neq 1}^k \frac{c_i}{c_1} t_i$$

Fourier-Motzkin Algorithm (1)

Goal: deal with inequalities in linear arithmetics

basic notions

- An inequality C in canonical form:

$$\sum_{i=1}^{n} a_i x_i \leq a_0 \qquad a_i \in \mathbb{Q}$$

- Note αC the multiplication of an inequation with a coefficient α:

$$\sum_{i=1}^{n} \alpha a_i x_i \leq \alpha a_0$$

- Note $C_1 + C_2$ the addition of two inequations :

$$\sum_{i=1}^{n} (a_i + b_i) x_i \leq a_0 + b_0$$

Fourier-Motzkin Algorithm (2)

Set $I = \{C_1 \cdots C_n\}$ the starting set of inequations. Each step of the algorithm will eliminate a variable from the set of the equations.

- Let I^+ (I^-) be the set of equations where x appears with positive (negative) coefficient
- Compute

$$I_x = \bigcup_{C \in I^-, D \in I^+} \beta C + \alpha D \qquad \alpha x \in C, -\beta x \in D$$

- Let I_0 the set of inequations in I without x
- Replace I par $I' = I_0 \cup I_x$
- In particular, if x appears only with coefficients of the same sign in I, suppress all inequations where x appears
- When I does not contain variables any more, either we have satisfiable inequalities (like $1 \leq 2$) or an inconsistency

Fourier-Motzkin Algorithm (3)

- Complexity: double exponential
- Not incremental
- Still behaves well in practice
- Can be easily extended to deduce equations between terms

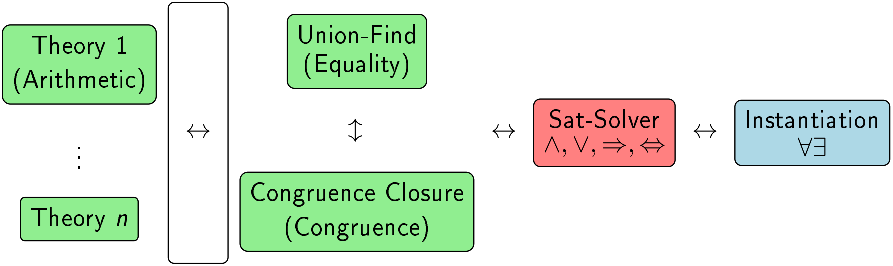
Simplex Algorithm (1947)

- Initially designed to solve the optimization problem:

$$\begin{array}{rl} \text{maximize} & \vec{c} \cdot \vec{x} \\ \text{subject to} & A\vec{x} \leq \vec{b} \quad \vec{x} \geq \vec{0} \end{array}$$

- Complexity: exponential (polynomial in practice)
- incremental variants exist

Combination of Theories



Combination of Theories

We have seen different decision procedures for different theories:

- The theory of uninterpreted function symbols (Congruence Closure)
- Linear arithmetics (Fourier-Motzkin, Simplex)

Other decision procedures exist:

- For the theory of pairs (or lists): `nil` and `cons`
- The theory of arrays: `get` and `set`
- The theory of bitvectors: `get`, `append`, `shift`

How do they work together?

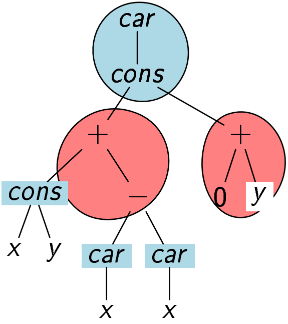
Nelson Oppen Combination — 1980 (1)

Consider a term with symbols from multiple theories:

$$car(cons(cons(x, y) + (car(x) - car(x)), y + 0))$$

Variable abstraction

Introduce variables to build pure terms and equations between variables:



$$\begin{aligned} z_1 &= car(x) \\ z_2 &= cons(x, y) \\ z_3 &= z_2 + (z_1 - z_1) \\ z_4 &= y + 0 \\ z_5 &= car(cons(z_3, z_4)) \end{aligned}$$

Nelson Oppen Combination (2)

First step

Transform the problem into n pure problems and a set of equations between auxiliary variables, using variable abstraction

Second step

- Run each decision procedure independently on its pure equations
- Communicate newly discovered equalities to the other decision procedures

Shostak Combination — 1978 (1)

Characteristics

- avoids costly equality propagation in Nelson-Oppen
- Requires a solver instead of a decision procedure: a solver takes an equation $t_1 = t_2$ and returns a substitution $x_i \mapsto t_i$
- Requires a canonizer, putting terms of the theory in normal form (sum of monomials, ...)
- Restriction: applies only to theories of equality (inequalities in linear arithmetic not well supported)

Shostak Combination (2)

- Maintain a table mapping terms to their representatives (for example using an E-Graph with Union-find)

Terms	Representatives
t_1	r_1
t_2	r_1
t_3	r_2

- When treating an equation $a = b$, call the solver on a and b, obtain a substitution σ and apply this substitution to all representatives. Finally, canonize all representatives.

Shostak vs. Nelson-Oppen

Comparison

- Shostak supposedly faster (is it in practice?)
- It's harder to write a solver than it is to write a decision procedure
- Shostak works well in the case of Congruence Closure + other theory
- It is still active research to combine more than just one theory with CC in the Shostak framework
- Most solvers use a Nelson-Oppen like Approach
- Alt-Ergo uses a modified Shostak-like Approach

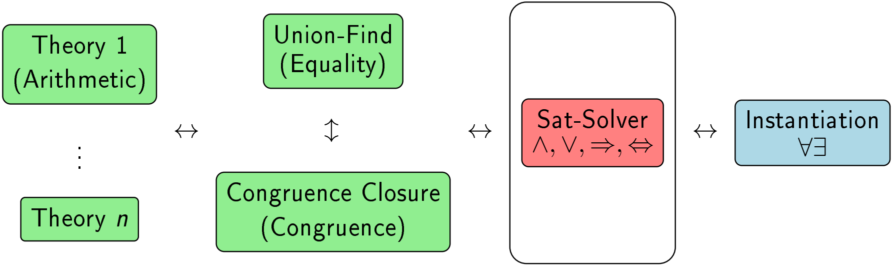
Theories in SMT solvers: Summary

- SMT solvers use Congruence Closure and various decision procedures for the “Theory part” of SMT
- SMT solvers are generally complete and terminating for conjunctions of atoms in this fragment of the theory
- If they don’t prove the goal, it’s wrong!

Beware

- non-linear arithmetic is undecidable
- on integer arithmetics, many provers use incomplete (but faster) heuristics

Satisfiability



Satisfiability

We have seen ...

- How to decide if a conjunction of equalities and inequalities is inconsistent

But what about the other connectors?

- Negation, disjunction, implication, equivalence

This is the work of a SAT-solver

A new input language

Atoms

- Formulas whose structure a SAT-solver will not look at
- Typically, equalities / inequalities, predicates
- Examples: $x = 0$, $x \leq f(x) + 3$, $even(t)$

Literal

An atom, with or without negation sign $\neg$

Conjunctive Normal form (CNF)

- A formula in CNF is a conjunction of disjunctions:

$$(a \vee b \vee \neg b) \wedge (b \vee c) \wedge (\neg c \vee d \vee e)$$

- The disjunctions are called clauses
- This is the input language of a SAT solver

Sat Solvers

Sat Solvers

- Decide if a formula in CNF is satisfiable
- In a satisfiable formula one can assign a truth value to each atom so that the formula becomes true
- Such an assignment is called a model
- This problem is NP-complete

Putting formulas in CNF

We won't talk about it here, we just mention...

- That the naive way of doing it leads to a formula that is exponentially bigger than the original one
- There are smarter ways of doing it that produce a CNF linear in the size of the original formula, but the formula is still much bigger
- One can do the conversion lazily, during the execution of the SAT algorithm (makes the algorithm more complicated)

DPLL (1)

The DPLL method

- Invented by Davis, Putnam (1960) and Davis, Logemann, Loveland(1962)
- Is an intelligent exhaustive search of a model for the input formula

A Unit Clause

- Is a clause which consists of a single literal
- This literal must be true if the formula is satisfiable

DPLL (2)

Boolean Constraint Propagation (BCP)

- Search for a unit clause and assume the corresponding literal l to be true
- Each literal of the form $\neg l$ is removed
- Each clause which contains l (positively) is removed
- Repeat

Case splitting

- Choose an arbitrary literal l occurring in the formula
- Try to find a model where l is true
- If there is none, try to find a model where $\neg l$ is true (backtracking)
- If there is none either, the formula is unsatisfiable

DPLL (3)

The algorithm

- As long as there are unit clauses, do BCP (simplify current formula)
- If there are no unit clauses, do a case split
- if the current formula contains the empty clause, it is unsatisfiable; backtrack

The algorithm as inference rules

$$\text{Axiom}\ \frac{}{\Gamma \vdash \Delta, \emptyset} \qquad \text{Unit}\ \frac{\Gamma, l \vdash \Delta}{\Gamma \vdash \Delta, l} \qquad \text{Elim}\ \frac{\Gamma, l \vdash \Delta}{\Gamma, l \vdash \Delta, l \ \vee \ C}$$

$$\text{Reduce}\ \frac{\Gamma \vdash \Delta, C}{\Gamma, l \vdash \Delta, \neg l \ \vee \ C} \qquad \text{Split}\ \frac{\Gamma, l \vdash \Delta \qquad \Gamma, \neg l \vdash \Delta}{\Gamma \vdash \Delta}$$

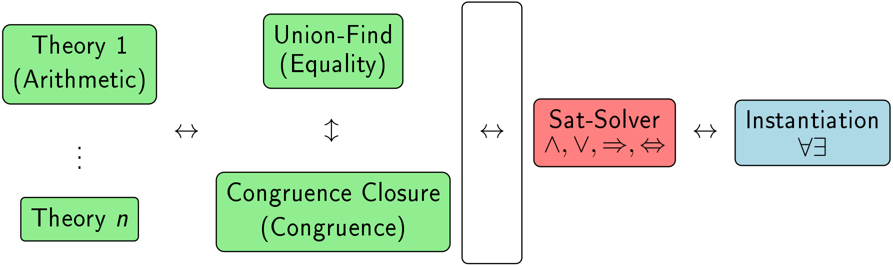
DPLL(4)

- DPLL is correct and complete
- Choice of the splitting variable is very important
- There are many heuristics to choose it

Check for the validity of a formula using DPLL

- Negate the input formula
- Search for a model of this formula (DPLL)
- If there is none, the formula is valid
- Otherwise the discovered model is a counterexample

Combining SAT and the theories



SAT and the Theory working together

The General Idea

- SAT decomposes the logical structure of the formula
- Each assumed literal is given to the appropriate theory
- which one? consider $t.[i-1] = x + y$
- A conflict can be detected in two ways:
  - The SAT: We have found the empty clause
  - The Theory: We have derived an absurd statement in some (combination of) theories

$$x = x + 1$$

DPLL — Backjumping

Optimisation

- Reconsider the Split rule:

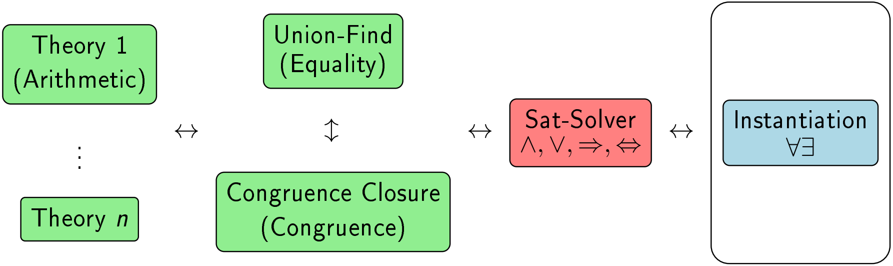
$$\text{Split}\ \frac{\Gamma, l \vdash \Delta \qquad \Gamma, \neg l \vdash \Delta}{\Gamma \vdash \Delta}$$

- Backtracking to $\neg l$ not always needed!
- When l is not involved in the inconsistency in the left branch, then one does not need to check the branch of $\neg l$

Collaboration needed

- The SAT must be able to explain conflicts
- The Theory must be able to explain conflicts

Instantiation



Dealing with quantifiers

Example

Axiom $A : \forall x : int.f(g(x), 0) = 0$

- Potentially, for any term t of integer type, add $f(g(t), 0) = 0$ to the knowledge base
- Where do the terms t come from?
- Usually, the nodes of the E-graph are considered (the set of ground terms)
- This is incomplete (a proof may require to invent a new term)
- Problem: taking all terms of integer type to instantiate axiom A is too much

Triggers (1)

Restricting the set of terms to consider

- Idea: do not allow instantiation of all terms of the right type
- Add a trigger, a term of a certain form to the axiom:
- Axiom $A : \forall x : int[g(x)].f(g(x), 0) = 0$
- Instantiate x only with terms t for which ground term $g(t)$ exists
- This is an heuristics (even more incomplete)
- Triggers can be given by the user, or computed by the prover

Triggers (2)

Matching

- Given a term t with variables and a ground term t', try to find a substitution σ for the variables in t such that $t\sigma = t'$
- This substitution is used to instantiate the entire axiom

Considerations

- A trigger can consist of several terms
- The trigger term(s) should contain all variables of the axiom
- The process of comparing the ground terms with the trigger is called matching
- A large trigger is more restrictive than a small trigger: We instantiate less (which is good), but we may miss necessary instantiations
- A variable as trigger is as good (as bad) as no trigger

E-Matching

Use the E-Graph to enhance matching

- Use the equality information of the E-graph to match more terms
- Example: The E-graph of $f(a) = a$ actually describes also $f(f(a))$ and $f(f(f(a)))$ ...

- A trigger of the form $f(a)$ can be instantiated by any $f^n(a)$

Skolemization

What to do with existential quantifiers?

- We can eliminate them using Skolemization
- An axiom $\exists x.P(x)$ becomes $P(c)$ for a fresh Skolem constant c
- If quantifiers are nested:

$$\forall x \exists y.P(x, y)$$

  becomes

$$\forall x.P(x, f(x))$$

  for a fresh Skolem function f

The Big Picture (1)

- Take a first-order formula as input
- Negate it — try to prove that the negation is unsatisfiable
- Eliminate Existentials (Skolemization)
- Compute triggers (if not given) for quantifiers
- Put it in CNF (or do it lazily)
- Run the SAT solver on the CNF
- Every assumed atom is given to the theories
- Inconsistencies can be detected by
  - the SAT: an empty clause present
  - the theory: set of assumed literals is inconsistent wrt. the theory
- At some point (when? how often?), do Matching to instantiate axioms, obtain new terms and new facts

The Big Picture (2)

- SMT provers are generally complete over the SAT + theory part (with restrictions)
- SMT provers are incomplete over the matching/instantiation part

SMTLib, SMTComp

- There is a common input format accepted by most SMT solvers: SMTLib
- SMT provers: Alt-Ergo, Z3, Yices, CVC3 (all with quantifiers and theories), MathSat, Boolector (without quantifiers)
- There is an annual competition with many different categories: with/without quantifiers, arithmetics, bit-vectors, arrays,...
- 2009 winner: mostly Yices (in categories without quantifiers) and CVC3 (in categories with quantifiers)
- in a few categories, 2008 winner Z3 was actually faster (but not officially part of 2009 competition)

Resolution based provers (1)

Unification

Generalization of Matching: Given two terms t_1 and t_2 both containing variables, find a substitution σ such that $t_1\sigma = t_2\sigma$.

Resolution rule

$$\text{Resolution}\ \frac{C_1 \ \vee \ l_1 \qquad C_2 \ \vee \ \neg l_2 \qquad \sigma = \mathit{unif}(l_1, l_2)}{C_1\sigma \ \vee \ C_2\sigma}$$

Example

$$\text{Resolution}\ \frac{\neg P(x) \ \vee \ Q(x) \qquad P(a) \qquad \sigma = [x \mapsto a]}{Q(a)}$$

Resolution based provers (2)

Algorithm

- Just as SMT solvers, negate formula, skolemize and put in CNF
- Now, using the Resolution rule, derive new clauses
- When encountering the empty clause, the initial formula was valid

TPTP Competition

- Participants: Vampire, Spass, E, ...
- Recent winners of various categories: Vampire, Waldmeister

Resolution based provers (3) - Comparison

Advantages

- Resolution is complete for first-order logic
- It implements a semi-decision procedure:
  - If the formula is valid, the algorithm eventually stops, with the right answer
  - If the formula is invalid, it may run forever
- Resolution is much better at proving complex theorems without arithmetic reasoning

Disadvantages

- Integrating equality reasoning is difficult, but has been mastered: Superposition, Paramodulation
- Integration of arithmetic or other theories is currently out of reach of the state of the art

Tableau based provers (Zenon)

Tableau method

- Do not transform formula into CNF, but do negate it
- Apply rules to formula to build a refutation tree
- Rules for conjunction and disjunction:

$$\begin{array}{c} A \wedge B \\ | \\ A \\ | \\ B \end{array} \qquad\qquad \begin{array}{c} A \vee B \\ / \quad \backslash \\ A \quad B \end{array}$$

- If a branch contains a contradiction (A and $\neg A$, $x \neq x$, False, $\ldots$), it is closed
- When all branches are closed, the negated formula is unsatisfiable
- Prominent prover in this class: Zenon
- It is the only prover that produces a machine-checkable proof
- Tableau method generally considered to be slower than

Conclusion

SMT provers . . .

- Divide the input formula into a theory part, a propositional part, and lemmas
- Are complete wrt. the propositional part, and selected theories
- Are incomplete in general
- The combination of decision procedures, but also the combination of the SAT solver and the theories are the main challenges of SMT